An Integral Containing the Square of a Bessel Function

By J. N. Newman and W. Frank

In the process of investigating the hydrodynamical characteristics of floating and submerged ellipsoids, an integral arose of the form

$$I_n^m(x) = \int_0^{\pi/2} \frac{J_n^2(x \cos \theta)}{(x \cos \theta)^{2m}}\, d\theta \tag{1}$$

where m and n are either integers, or integers plus one half, $0 \leqq m \leqq n$, and J_n is the Bessel function of the first kind of order n. For the case where m and n are integers plus one half, the integral (1) can be conveniently expressed in terms of the spherical Bessel function $j_n(z) = \sqrt{\frac{\pi}{2z}}\, J_{n+1/2}(z)$. Then

$$I_{n+1/2}^{m+1/2}(x) = \frac{2}{\pi} \int_0^{\pi/2} \frac{j_n^2(x \cos \theta)}{(x \cos \theta)^{2m}}\, d\theta. \tag{1'}$$

Since no reference could be found to the properties of the above integrals, we present here certain of their properties and a short table of numerical values. A more extensive table for integer values of m and n, giving values to 5S, for $x = .1(.1)10.0$, $n = 0(1)9$, and $m = 0(1)n$, is on deposit in the Unpublished Mathematical Tables file.

The integral (1) can be expressed as a generalized hypergeometric function ${}_2F_3$ by using the Neumann series [1] for J_n^2 and interchanging the orders of summation and integration. In this manner we obtain

$$\begin{aligned} I_n^m(x) &= \frac{1}{2} \sum_{K=0}^{\infty} \frac{(-1)^k x^{2(K+n-m)} \Gamma(n+K+\frac{1}{2}) \Gamma(n+K-m+\frac{1}{2})}{K!\Gamma(n+K+1)\Gamma(2n+K+1)\Gamma(K+n-m+1)} \\ &= \frac{\sqrt{\pi}}{2^{2n+1}} \frac{\Gamma(n-m+\frac{1}{2})}{[\Gamma(n+1)]^2 \Gamma(n-m+1)}\, x^{2n-2m} \\ &\qquad {}_2F_3\left(\begin{matrix} n+\frac{1}{2}, n-m+\frac{1}{2}; -x^2 \\ n+1, 2n+1, n-m+1 \end{matrix} \right) \end{aligned} \tag{2}$$

This series expansion is everywhere convergent and provides an efficient means for computing decimal approximations to the integrals (1) unless x is very large. For large x, asymptotic representations can be derived as follows. Changing the variable of integration to $y = x \cos \theta$, it follows that

$$\begin{aligned} I_n^m(x) &= \int_0^x \frac{J_n^2(y)}{y^{2m}\sqrt{x^2 - y^2}}\, dy \\ &= \int_0^{x^{1/2}} \frac{J_n^2(y)}{y^{2m}\sqrt{x^2 - y^2}}\, dy + \int_{x^{1/2}}^{x} \frac{J_n^2(y)}{y^{2m}\sqrt{x^2 - y^2}}\, dy. \end{aligned} \tag{3}$$

Received April 20, 1962.

For $m \neq 0$ the contribution from the first interval is

$$(4)\qquad \begin{aligned}\int_0^{x^{1/2}} \frac{J_n^2(y)}{y^{2m}\sqrt{x^2-y^2}}\,dy &\cong \frac{1}{x}\int_0^{x^{1/2}} \frac{J_n^2(y)}{y^{2m}}\,dy \\ &\cong \frac{1}{x}\int_0^{\infty} \frac{J_n^2(y)}{y^{2m}}\,dy = \frac{1}{x}\,\frac{\Gamma(m)\Gamma(n-m+\frac{1}{2})}{2\sqrt{\pi}\,\Gamma(m+\frac{1}{2})\Gamma(n+m+\frac{1}{2})}\end{aligned}$$

and the contribution from the second interval is small,* of order $x^{-(m+1)}\log x$. For $m = 0$, the contribution from the first interval is

$$(5)\qquad \begin{aligned}\int_0^{x^{1/2}} \frac{J_n^2(y)}{\sqrt{x^2-y^2}}\,dy &\cong \frac{1}{x}\int_0^{x^{1/2}} J_n^2(y)\,dy \\ &= \frac{1}{x}\int_0^{x^{1/2}} [J_n^2(y) - f(y)]\,dy + \frac{1}{x}\int_0^{x^{1/2}} f(y)\,dy\end{aligned}$$

where $f(y)$ is arbitrary. The asymptotic behavior of $J_n^2(y)$ suggests setting

$$(6)\qquad f(y) = \frac{1}{\pi(y+1)},$$

for, with such a choice,

$$(7)\qquad \begin{aligned}\int_0^{x^{1/2}} [J_n^2(y) - f(y)]\,dy &\cong \int_0^{\infty} [J_n^2(y) - f(y)]\,dy \\ &= \lim_{p\to 0}\int_0^{\infty} e^{-py}[J_n^2(y) - f(y)]\,dy \\ &= \frac{1}{\pi}\lim_{p\to 0}\,[Q_{n-1/2}(1+\tfrac{1}{2}p^2) + e^{p}Ei(-p)] \\ &= \frac{1}{\pi}\lim_{p\to 0}\,[-\gamma - \psi(n+\tfrac{1}{2}) - \log\tfrac{1}{2}p + \gamma + \log p] \\ &= \frac{1}{\pi}\,[\log 2 - \psi(n+\tfrac{1}{2})]\,.\end{aligned}$$

Here $Q_{n-1/2}$ is the Legendre function of the second kind, $Ei(-p)$ is the exponential integral, and ψ is the logarithmic derivative of the gamma function. Since

$$(8)\qquad \int_0^{x^{1/2}} f(y)\,dy = \frac{1}{\pi}\log(1+x^{1/2}) = \frac{1}{2\pi}\log x + 0(x^{-1/2}),$$

it follows that

$$(9)\qquad \int_0^{x^{1/2}} \frac{J_n^2(y)}{\sqrt{x^2-y^2}}\,dy = \frac{1}{\pi x}\,[\log 2x^{1/2} - \psi(n+\tfrac{1}{2})] + 0(x^{-3/2}).$$

---

* This follows by substituting the asymptotic expansion of J_n for large argument and intergrating term-by-term.

TABLE 1

The Integrals $I_n^m(x)$

x	$I_0^0(x)$	$I_1^0(x)$	$I_1^1(x)$	$I_2^0(x)$	$I_2^1(x)$
0.0	0.15708E 01	0.00000	0.39270E—00	0.00000	0.00000
0.2	0.15552E 01	0.77953E—02	0.39074E—00	0.14645E—04	0.48843E—03
0.4	0.15094E 01	0.30486E—01	0.38494E—00	0.23044E—03	0.19246E—02
0.6	0.14364E 01	0.66061E—01	0.37552E—00	0.11345E—02	0.42234E—02
0.8	0.13410E 01	0.11140E—00	0.36281E—00	0.34482E—02	0.72495E—02
1.0	0.12293E 01	0.16257E—00	0.34727E—00	0.80041E—02	0.10828E—01
1.2	0.11079E 01	0.21528E—00	0.32943E—00	0.15600E—01	0.14754E—01
1.4	0.98399E 00	0.26522E—00	0.30990E—00	0.26850E—01	0.18813E—01
1.6	0.86398E 00	0.30856E—00	0.28929E—00	0.42051E—01	0.22786E—01
1.8	0.75351E 00	0.34224E—00	0.26820E—00	0.61095E—01	0.26473E—01
2.0	0.65689E 00	0.36425E—00	0.24721E—00	0.83422E—01	0.29700E—01
2.2	0.57682E 00	0.37379E—00	0.22682E—00	0.10804E—00	0.32330E—01
2.4	0.51438E 00	0.37123E—00	0.20747E—00	0.13361E—00	0.34271E—01
2.6	0.46907E—00	0.35804E—00	0.18949E—00	0.15857E—00	0.35478E—01
2.8	0.43902E—00	0.33651E—00	0.17311E—00	0.18127E—00	0.35950E—01
3.0	0.42139E—00	0.30950E—00	0.15847E—00	0.20020E—00	0.35734E—01
3.2	0.41273E—00	0.28008E—00	0.14560E—00	0.21413E—00	0.34908E—01
3.4	0.40944E—00	0.25117E—00	0.13445E—00	0.22223E—00	0.33579E—01
3.6	0.40814E—00	0.22526E—00	0.12493E—00	0.22419E—00	0.31872E—01
3.8	0.40602E—00	0.20420E—00	0.11685E—00	0.22027E—00	0.29917E—01
4.0	0.40106E—00	0.18903E—00	0.11003E—00	0.21121E—00	0.27842E—01
4.2	0.39217E—00	0.17997E—00	0.10427E—00	0.19821E—00	0.25762E—01
4.4	0.37910E—00	0.17649E—00	0.99342E—01	0.18276E—00	0.23774E—01
4.6	0.36243E—00	0.17745E—00	0.95069E—01	0.16644E—00	0.21952E—01
4.8	0.34331E—00	0.18132E—00	0.91279E—01	0.15083E—00	0.20345E—01
5.0	0.32321E—00	0.18641E—00	0.87835E—01	0.13726E—00	0.18977E—01
5.2	0.30371E—00	0.19110E—00	0.84634E—01	0.12671E—00	0.17849E—01
5.4	0.28619E—00	0.19407E—00	0.81603E—01	0.11973E—00	0.16943E—01
5.6	0.27172E—00	0.19439E—00	0.78700E—01	0.11642E—00	0.16230E—01
5.8	0.26088E—00	0.19167E—00	0.75904E—01	0.11640E—00	0.15670E—01
6.0	0.25373E—00	0.18600E—00	0.73214E—01	0.11896E—00	0.15220E—01
6.2	0.24986E—00	0.17792E—00	0.70639E—01	0.12312E—00	0.14839E—01
6.4	0.24848E—00	0.16830E—00	0.68193E—01	0.12781E—00	0.14491E—01
6.6	0.24857E—00	0.15817E—00	0.65890E—01	0.13199E—00	0.14150E—01
6.8	0.24902E—00	0.14856E—00	0.63741E—01	0.13480E—00	0.13795E—01
7.0	0.24883E—00	0.14039E—00	0.61751E—01	0.13565E—00	0.13419E—01
7.2	0.24722E—00	0.13427E—00	0.59916E—01	0.13426E—00	0.13019E—01
7.4	0.24372E—00	0.13052E—00	0.58227E—01	0.13072E—00	0.12602E—01
7.6	0.23824E—00	0.12909E—00	0.56669E—01	0.12539E—00	0.12177E—01
7.8	0.23100E—00	0.12961E—00	0.55225E—01	0.11888E—00	0.11756E—01
8.0	0.22254E—00	0.13147E—00	0.53873E—01	0.11194E—00	0.11352E—01
8.2	0.21354E—00	0.13392E—00	0.52596E—01	0.10532E—00	0.10976E—01
8.4	0.20476E—00	0.13619E—00	0.51377E—01	0.99686E—01	0.10634E—01
8.6	0.19689E—00	0.13761E—00	0.50201E—01	0.95535E—01	0.10329E—01
8.8	0.19047E—00	0.13770E—00	0.49062E—01	0.93123E—01	0.10063E—01
9.0	0.18580E—00	0.13620E—00	0.47953E—01	0.92452E—01	0.98310E—02
9.2	0.18293E—00	0.13315E—00	0.46874E—01	0.93280E—01	0.96272E—02
9.4	0.18165E—00	0.12879E—00	0.45827E—01	0.95173E—01	0.94440E—02
9.6	0.18154E—00	0.12359E—00	0.44815E—01	0.97577E—01	0.92736E—02
9.8	0.18204E—00	0.11810E—00	0.43843E—01	0.99905E—01	0.91088E—02
10.0	0.18257E—00	0.11290E—00	0.42913E—01	0.10162E—00	0.89440E—02

x	$I_2^2(x)$	$I_3^0(x)$	$I_3^1(x)$	$I_3^2(x)$	$I_3^3(x)$
0.0	0.24544E—01	0.00000	0.00000	0.00000	0.68177E—03
0.2	0.24462E—01	0.13576E—07	0.40736E—06	0.13584E—04	0.68007E—03
0.4	0.24219E—01	0.85752E—06	0.64368E—05	0.53730E—04	0.67500E—03
0.6	0.23822E—01	0.95559E—05	0.31913E—04	0.11865E—03	0.66666E—03
0.8	0.23279E—01	0.52065E—04	0.97954E—04	0.20546E—03	0.65522E—03
1.0	0.22606E—01	0.19089E—03	0.23030E—03	0.31038E—03	0.64091E—03
1.2	0.21819E—01	0.54296E—03	0.45602E—03	0.42889E—03	0.62399E—03
1.4	0.20936E—01	0.12924E—02	0.79987E—03	0.55601E—03	0.60479E—03
1.6	0.19978E—01	0.26935E—02	0.12808E—02	0.68654E—03	0.58366E—03

TABLE 1 (*Continued*)

x	$I_2^2(x)$	$I_3^0(x)$	$I_3^1(x)$	$I_3^2(x)$	$I_3^3(x)$
1.8	0.18966E—01	0.50600E—02	0.19091E—02	0.81535E—03	0.56096E—03
2.0	0.17920E—01	0.87397E—02	0.26840E—02	0.93759E—03	0.53708E—03
2.2	0.16862E—01	0.14075E—01	0.35927E—02	0.10490E—02	0.51241E—03
2.4	0.15810E—01	0.21356E—01	0.46103E—02	0.11459E—02	0.48730E—03
2.6	0.14780E—01	0.30763E—01	0.57014E—02	0.12257E—02	0.46211E—03
2.8	0.13787E—01	0.42323E—01	0.68224E—02	0.12864E—02	0.43716E—03
3.0	0.12844E—01	0.55872E—01	0.79243E—02	0.13274E—02	0.41274E—03
3.2	0.11958E—01	0.71032E—01	0.89571E—02	0.13487E—02	0.38910E—03
3.4	0.11136E—01	0.87224E—01	0.98734E—02	0.13511E—02	0.36643E—03
3.6	0.10382E—01	0.10369E—00	0.10632E—01	0.13363E—02	0.34491E—03
3.8	0.96968E—02	0.11956E—00	0.11202E—01	0.13066E—02	0.32465E—03
4.0	0.90789E—02	0.13393E—00	0.11564E—01	0.12645E—02	0.30572E—03
4.2	0.85258E—02	0.14593E—00	0.11710E—01	0.12129E—02	0.28816E—03
4.4	0.80331E—02	0.15482E—00	0.11649E—01	0.11545E—02	0.27198E—03
4.6	0.75958E—02	0.16012E—00	0.11399E—01	0.10923E—02	0.25715E—03
4.8	0.72080E—02	0.16157E—00	0.10989E—01	0.10286E—02	0.24361E—03
5.0	0.68638E—02	0.15927E—00	0.10456E—01	0.96574E—03	0.23130E—03
5.2	0.65573E—02	0.15361E—00	0.98409E—02	0.90540E—03	0.22013E—03
5.4	0.62827E—02	0.14525E—00	0.91847E—02	0.84894E—03	0.21000E—03
5.6	0.60352E—02	0.13508E—00	0.85267E—02	0.79725E—03	0.20082E—03
5.8	0.58101E—02	0.12411E—00	0.79006E—02	0.75082E—03	0.19249E—03
6.0	0.56037E-02	0.11334E-00	0.73330E-02	0.70975E-03	0.18492E-03
6.2	0.54129E—02	0.10371E—00	0.68417E—02	0.67386E—03	0.17800E—03
6.4	0.52351E—02	0.95967E—01	0.64357E—02	0.64272E—03	0.17167E—03
6.6	0.50685E—02	0.90596E—01	0.61153E—02	0.61577E—03	0.16584E—03
6.8	0.49117E—02	0.87792E—01	0.58738E—02	0.59235E—03	0.16044E—03
7.0	0.47635E—02	0.87441E—01	0.56986E—02	0.57180E—03	0.15544E—03
7.2	0.46234E—02	0.89158E—01	0.55738E—02	0.55347E—03	0.15076E—03
7.4	0.44907E—02	0.92342E—01	0.54820E—02	0.53682E—03	0.14638E—03
7.6	0.43651E—02	0.96263E—01	0.54065E—02	0.52136E—03	0.14225E—03
7.8	0.42461E—02	0.10016E—00	0.53328E—02	0.50677E—03	0.13836E—03
8.0	0.41336E—02	0.10332E—00	0.52495E—02	0.49278E—03	0.13468E—03
8.2	0.40271E—02	0.10520E—00	0.51499E—02	0.47926E—03	0.13119E—03
8.4	0.39263E—02	0.10546E—00	0.50310E—02	0.46615E—03	0.12788E—03
8.6	0.38308E—02	0.10398E—00	0.48938E—02	0.45343E—03	0.12473E—03
8.8	0.37403E—02	0.10090E—00	0.47424E—02	0.44115E—03	0.12174E—03
9.0	0.36543E—02	0.96576E—01	0.45830E—02	0.42935E—03	0.11889E—03
9.2	0.35724E—02	0.91495E—01	0.44227E—02	0.41811E—03	0.11617E—03
9.4	0.34943E—02	0.86235E—01	0.42685E—02	0.40744E—03	0.11357E—03
9.6	0.34196E—02	0.81365E—01	0.41261E—02	0.39740E—03	0.11110E—03
9.8	0.33481E—02	0.77375E—01	0.39999E—02	0.38796E—03	0.10873E—03
10.0	0.32795E—02	0.74610E—01	0.38920E—02	0.37911E—03	0.10647E—03

The contribution from the second interval in (3), with $m = 0$, is

$$\int_{x^{1/2}}^{x} \frac{J_n^2(y)}{\sqrt{x^2 - y^2}}\, dy \cong \frac{2}{\pi} \int_{x^{1/2}}^{x} \frac{\cos^2\left(y - \frac{\pi}{2} n - \frac{\pi}{4}\right)}{y\sqrt{x^2 - y^2}}\, dy$$

(10)
$$\cong \frac{1}{\pi} \int_{x^{1/2}}^{x} \frac{dy}{y\sqrt{x^2 - y^2}} = \frac{1}{\pi x} \log\,(x^{1/2} + \sqrt{x - 1})$$

$$= \frac{1}{\pi x} \log 2x^{1/2} + 0(x^{-3/2}).$$

Thus, combining these results, we have the following asymptotic representations

(11) $$I_n^0(x) \sim \frac{1}{\pi x} [\log 4x - \psi(n + \tfrac{1}{2})]$$

(12) $$I_n^m(x) \sim \frac{1}{2\sqrt{\pi x}} \frac{\Gamma(m)}{\Gamma(m + \frac{1}{2})} \frac{\Gamma(n - m + \frac{1}{2})}{\Gamma(n + m + \frac{1}{2})} \qquad (m = 1, 2, \cdots n).$$

The numerical evaluation of $I_n^m(x)$ was performed on an IBM 7090 system, using two different methods. The first method consisted of evaluating the integral (1) by means of the Gauss-Mehler quadrature formula, whence

$$I_n^m(x) \cong \frac{\pi}{K} \sum_{k=1}^{K/2} \frac{J_n^2\left(x \cos \frac{2k-1}{2K}\pi\right)}{\left(x \cos \frac{2k-1}{2K}\pi\right)^{2m}} \tag{13}$$

TABLE 2

The Integrals $I_{n+\frac{1}{2}}^{m+\frac{1}{2}}(x)$

x	$I_{\frac{1}{2}}^{\frac{1}{2}}(x)$	$I_{\frac{3}{2}}^{\frac{1}{2}}(x)$	$I_{\frac{3}{2}}^{\frac{3}{2}}(x)$	$I_{\frac{5}{2}}^{\frac{1}{2}}(x)$	$I_{\frac{5}{2}}^{\frac{3}{2}}(x)$
0.0	0.10000E 01	0.00000	0.11111E—00	0.00000	0.00000
0.2	0.99336E 00	0.22089E—02	0.11067E—00	0.26540E—05	0.88509E—04
0.4	0.97376E 00	0.86780E—02	0.10935E—00	0.41861E—04	0.34951E—03
0.6	0.94211E 00	0.18947E—01	0.10720E—00	0.20693E—03	0.76972E—03
0.8	0.89991E 00	0.32294E—01	0.10429E—00	0.63246E—03	0.13279E—02
1.0	0.84905E 00	0.47792E—01	0.10069E—00	0.14789E—02	0.19963E—02
1.2	0.79176E 00	0.64391E—01	0.96508E—01	0.29087E—02	0.27422E—02
1.4	0.73043E 00	0.80999E—01	0.91868E—01	0.50610E—02	0.35299E—02
1.6	0.66746E 00	0.96569E—01	0.86889E—01	0.80283E—02	0.43229E—02
1.8	0.60512E 00	0.11018E—00	0.81696E—01	0.11838E—01	0.50863E—02
2.0	0.54539E 00	0.12111E—00	0.76411E—01	0.16439E—01	0.57880E—02
2.2	0.48991E—00	0.12886E—00	0.71149E—01	0.21702E—01	0.64008E—02
2.4	0.43988E—00	0.13319E—00	0.66011E—01	0.27422E—01	0.69039E—02
2.6	0.39604E—00	0.13415E—00	0.61084E—01	0.33336E—01	0.72831E—02
2.8	0.35866E—00	0.13200E—00	0.56438E—01	0.39143E—01	0.75316E—02
3.0	0.32764E—00	0.12720E—00	0.52125E—01	0.44531E—01	0.76494E—02
3.2	0.30248E—00	0.12036E—00	0.48179E—01	0.49206E—01	0.76432E—02
3.4	0.28245E—00	0.11216E—00	0.44614E—01	0.52919E—01	0.75251E—02
3.6	0.26667E—00	0.10329E—00	0.41432E—01	0.55489E—01	0.73115E—02
3.8	0.25418E—00	0.94407E—01	0.38619E—01	0.56818E—01	0.70216E—02
4.0	0.24403E—00	0.86051E—01	0.36152E—01	0.56898E—01	0.66760E—02
4.2	0.23538E—00	0.78638E—01	0.33999E—01	0.55814E—01	0.62954E—02
4.4	0.22755E—00	0.72432E—01	0.32126E—01	0.53729E—01	0.58992E—02
4.6	0.22001E—00	0.67540E—01	0.30492E—01	0.50869E—01	0.55047E—02
4.8	0.21245E—00	0.63928E—01	0.29062E—01	0.47498E—01	0.51259E—02
5.0	0.20471E—00	0.61446E—01	0.27798E—01	0.43895E—01	0.47738E—02
5.2	0.19679E—00	0.59859E—01	0.26668E—01	0.40326E—01	0.44556E—02
5.4	0.18880E—00	0.58890E—01	0.25647E—01	0.37021E—01	0.41750E—02
5.6	0.18092E—00	0.58253E—01	0.24710E—01	0.34158E—01	0.39329E—02
5.8	0.17334E—00	0.57690E—01	0.23841E—01	0.31850E—01	0.37274E—02
6.0	0.16626E—00	0.56993E—01	0.23027E—01	0.30140E—01	0.35548E—02
6.2	0.15982E—00	0.56024E—01	0.22258E—01	0.29007E—01	0.34102E—02
6.4	0.15409E—00	0.54717E—01	0.21530E—01	0.28374E—01	0.32879E—02
6.6	0.14909E—00	0.53076E—01	0.20840E—01	0.28123E—01	0.31824E—02
6.8	0.14478E—00	0.51163E—01	0.20185E—01	0.28115E—01	0.30886E—02
7.0	0.14104E—00	0.49083E—01	0.19565E—01	0.28205E—01	0.30022E—02
7.2	0.13776E—00	0.46958E—01	0.18980E—01	0.28262E—01	0.29199E—02
7.4	0.13477E—00	0.44913E—01	0.18429E—01	0.28182E—01	0.28395E—02
7.6	0.13194E—00	0.43054E—01	0.17912E—01	0.27896E—01	0.27598E—02
7.8	0.12914E—00	0.41458E—01	0.17426E—01	0.27376E—01	0.26804E—02
8.0	0.12631E—00	0.40163E—01	0.16970E—01	0.26631E—01	0.26016E—02
8.2	0.12339E—00	0.39169E—01	0.16541E—01	0.25704E—01	0.25242E—02
8.4	0.12038E—00	0.38440E—01	0.16136E—01	0.24661E—01	0.24491E—02
8.6	0.11731E—00	0.37912E—01	0.15754E—01	0.23579E—01	0.23774E—02
8.8	0.11425E—00	0.37508E—01	0.15391E—01	0.22536E—01	0.23097E—02
9.0	0.11126E—00	0.37147E—01	0.15044E—01	0.21601E—01	0.22467E—02
9.2	0.10840E—00	0.36755E—01	0.14712E—01	0.20821E—01	0.21885E—02
9.4	0.10573E—00	0.36277E—01	0.14394E—01	0.20224E—01	0.21351E—02
9.6	0.10330E—00	0.35680E—01	0.14087E—01	0.19810E—01	0.20861E—02
9.8	0.10111E—00	0.34956E—01	0.13792E—01	0.19558E—01	0.20409E—02
10.0	0.99155E—01	0.34120E—01	0.13508E—01	0.19427E—01	0.19987E—02

TABLE 2 (*Continued*)

x	$I_{5/2}^{5/2}(x)$	$I_{7/2}^{1/2}(x)$	$I_{7/2}^{3/2}(x)$	$I_{7/2}^{5/2}(x)$	$I_{7/2}^{7/2}(x)$
0.0	0.44444E—02	0.00000	0.00000	0.00000	0.90703E—04
0.2	0.44318E—02	0.18070E—08	0.54221E—07	0.18080E—05	0.90502E—04
0.4	0.43940E—02	0.11431E—06	0.85794E—06	0.71601E—05	0.89902E—04
0.6	0.43321E—02	0.12769E—05	0.42635E—05	0.15844E—04	0.88913E—04
0.8	0.42473E—02	0.69813E—05	0.13129E—04	0.27518E—04	0.87555E—04
1.0	0.41416E—02	0.25711E—04	0.31000E—04	0.41727E—04	0.85850E—04
1.2	0.40173E—02	0.73533E—04	0.61702E—04	0.57926E—04	0.83828E—04
1.4	0.38769E—02	0.17618E—03	0.10890E—03	0.75507E—04	0.81524E—04
1.6	0.37233E—02	0.37001E—03	0.17565E—03	0.93825E—04	0.78976E—04
1.8	0.35595E—02	0.70127E—03	0.26399E—03	0.11223E—03	0.76223E—04
2.0	0.33885E—02	0.12235E—02	0.37463E—03	0.13010E—03	0.73309E—04
2.2	0.32134E—02	0.19928E—02	0.50675E—03	0.14686E—03	0.70277E—04
2.4	0.30370E—02	0.30620E—02	0.65791E—03	0.16200E—03	0.67168E—04
2.6	0.28618E—02	0.44733E—02	0.82417E—03	0.17512E—03	0.64022E—04
2.8	0.26904E—02	0.62509E—02	0.10002E—02	0.18590E—03	0.60879E—04
3.0	0.25247E—02	0.83948E—02	0.11798E—02	0.19416E—03	0.57772E—04
3.2	0.23663E—02	0.10876E—01	0.13562E—02	0.19981E—03	0.54732E—04
3.4	0.22167E—02	0.13635E—01	0.15222E—02	0.20287E—03	0.51786E—04
3.6	0.20766E—02	0.16581E—01	0.16715E—02	0.20346E—03	0.48957E—04
3.8	0.19467E—02	0.19600E—01	0.17983E—02	0.20177E—03	0.46261E—04
4.0	0.18272E—02	0.22557E—01	0.18980E—02	0.19809E—03	0.43712E—04
4.2	0.17181E—02	0.25312E—01	0.19677E—02	0.19271E—03	0.41319E—04
4.4	0.16191E—02	0.27726E—01	0.20059E—02	0.18598E—03	0.39085E—04
4.6	0.15296E—02	0.29677E—01	0.20131E—02	0.17824E—03	0.37012E—04
4.8	0.14491E—02	0.31072E—01	0.19911E—02	0.16985E—03	0.35098E—04
5.0	0.13767E—02	0.31849E—01	0.19433E—02	0.16111E—03	0.33337E—04
5.2	0.13118E—02	0.31993E—01	0.18741E—02	0.15230E—03	0.31723E—04
5.4	0.12534E—02	0.31528E—01	0.17885E—02	0.14367E—03	0.30246E—04
5.6	0.12008E—02	0.30523E—01	0.16920E—02	0.13541E—03	0.28898E—04
5.8	0.11531E—02	0.29082E—01	0.15901E—02	0.12766E—03	0.27667E—04
6.0	0.11098E—02	0.27332E—01	0.14877E—02	0.12051E—03	0.26542E—04
6.2	0.10702E—02	0.25416E—01	0.13891E—02	0.11401E—03	0.25514E—04
6.4	0.10337E—02	0.23479E—01	0.12976E—02	0.10818E—03	0.24572E—04
6.6	0.99992E—03	0.21651E—01	0.12157E—02	0.10299E—03	0.23707E—04
6.8	0.96845E—03	0.20041E—01	0.11446E—02	0.98399E—04	0.22909E—04
7.0	0.93899E—03	0.18726E—01	0.10846E—02	0.94346E—04	0.22170E—04
7.2	0.91130E—03	0.17747E—01	0.10353E—02	0.90760E—04	0.21484E—04
7.4	0.88518E—03	0.17108E—01	0.99548E—03	0.87570E—04	0.20844E—04
7.6	0.86049E—03	0.16777E—01	0.96347E—03	0.84703E—04	0.20246E—04
7.8	0.83710E—03	0.16697E—01	0.93741E—03	0.82097E—04	0.19683E—04
8.0	0.81493E—03	0.16789E—01	0.91544E—03	0.79697E—04	0.19153E—04
8.2	0.79389E—03	0.16969E—01	0.89581E—03	0.77458E—04	0.18652E—04
8.4	0.77391E—03	0.17151E—01	0.87710E—03	0.75345E—04	0.18178E—04
8.6	0.75493E—03	0.17263E—01	0.85823E—03	0.73335E—04	0.17728E—04
8.8	0.73689E—03	0.17249E—01	0.83856E—03	0.71411E—04	0.17299E—04
9.0	0.71973E—03	0.17079E—01	0.81780E—03	0.69565E—04	0.16892E—04
9.2	0.70339E—03	0.16746E—01	0.79605E—03	0.67793E—04	0.16504E—04
9.4	0.68781E—03	0.16265E—01	0.77361E—03	0.66094E—04	0.16133E—04
9.6	0.67294E—03	0.15673E—01	0.75100E—03	0.64469E—04	0.15779E—04
9.8	0.65873E—03	0.15016E—01	0.72877E—03	0.62919E—04	0.15441E—04
10.0	0.64513E—03	0.14349E—01	0.70745E—03	0.61446E—04	0.15117E—04

where K is an even integer. In the program $K = 10$ was the initial value, this was iteratively doubled, and the resulting values of $I_n{}^m(x)$ compared with the previous values until the difference in all computed values of $I_n{}^m(x)$ was less than one part in 10,000. In the range of the computations, $K = 20$ was sufficient to give this accuracy. For the complete tabulation with integer values of m and n, that is, for $x = .1(.1)10.0$, $n = 0(1)9$, $m = 0(1)n$, this method required approximately 10 minutes of computing time on the IBM 7090. However, for $x \leqq .7$ this procedure broke down for large m, presumably owing to the very small values of the denominator. To overcome this difficulty, the series expression (2) was used, employing double-precision arithmetic to overcome the round-off error which occurred at large x. The series were truncated when the last term failed to change the sum by one part in 2^{-27}. This method required approximately four minutes of 7090 time for the large table, and was thus more satisfactory in all respects. Comparison of the numerical results from these two methods for $x > .7$ showed complete agreement to 5S with the exception of a very few scattered differences of a unit in the fifth figure. Since the use of the series seems likely to be the more accurate of the two methods, it is felt that the final results shown in Tables 1 and 2 can be considered as being accurate to 5 significant figures.

Hydromechanics Laboratory
David Taylor Model Basin
Washington 7, D. C.

1. G. N. WATSON, *A Treatise on the Theory of Bessel Functions*, Cambridge 1958, paragraph 2.61, p. 32.